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A mean-field formulation for the mean-variance control of discrete-time linear systems with multiplicative noises

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ABSTRACT
This paper considers the stochastic optimal control of a multi-period mean-variance trade-off performance criterion with and without constraints for discrete-time linear systems subject to multiplicative noises. We adopt a mean-field approach to tackle the problem and obtain a solution for the unconstrained case in terms of a Riccati-like difference equation. From this general result, we obtain a sufficient condition for a closed-form solution for one of the constrained problems considered in the paper. When particularised to the portfolio selection problem, we show that our results retrieve some of the results available in the literature. We conclude the paper by illustrating the obtained optimal controls with a multi-period portfolio selection problem where we minimise the sum of the mean-variance trade-off costs of a portfolio against a benchmark along the time.

1. Introduction

Lately, there has been an increasing interest in the literature for linear systems with multiplicative noises. This class of models, sometimes also combined with Markov jumps, has found many applications such as in signal processing systems (Gershon et al., 2001; Wang, 2002), biological motor systems (Yurchenkov, 2018), aerospace systems (Basin, Ramirez, 2018; Basin, Yu, 2018) and finance (Barbieri & Costa, 2018; Costa & de Paulo, 2008; Sun et al., 2019; Zhu et al., 2004). We can refer the book (Dragan et al., 2013) and references therein for a general overview for this class of models.

From the portfolio optimisation point of view, one of the main applications of linear systems with multiplicative noises is related to the classical portfolio's mean-variance (MV) problem. For a pension fund, for instance, the main goals of these problems necessarily consider restrictions such as maximise the expected return for a given level of risk or minimise the expected risk for a given level of expected return. This class of problems has been analysed for the continuous-time case (see for instance Zhou & Li, 2000) as well as for the discrete-time case, see for instance (Barbieri & Costa, 2018; Costa & Nabholz, 2007; Guo et al., 2012; Leippold et al., 2004; Li & Ng, 2000; Zhu et al., 2004). As pointed out in Li and Ng (2000), one of the difficulties associated to the multi-period MV portfolio optimisation problem is that it is nonseparable in the sense of dynamic programming due to the quadratic term that arises from the variance. One approach to overcome this difficulty considers an analogous auxiliary problem with no quadratic term that solves the unconstrained and some constrained problems (Li & Ng, 2000; Zhou & Li, 2000). There are other methods to overcome the nonseparability issue in the MV problem as presented in Cerný and Kallsen (2009), Li et al. (2002), Schweizer (1996) and Xia and Yan (2006). Following a different approach, the authors in Cui et al. (2014) obtained an optimal policy to the multi-period MV problem by using the mean-field formulation, which consists of directly tackling the nonseparable dynamic MV problem by including the expected values of the portfolio and the investment strategy as variables of the dynamical system.

In this paper, we generalise the scalar unified framework in Cui et al. (2014) for linear systems with multiplicative noises and use the mean-field approach to solve some classical MV problems with restrictions by reducing them to unconstrained problems. In general,
the mean-field type of optimal stochastic control models considers the states as well as their expected values in either the system dynamics or the objective functions, or both. The mean-field approach was introduced in physics by Kac (1956) who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. There are several results related to mean-field linear-quadratic (LQ) unconstrained problems applied to linear systems with multiplicative noises. See (Huang et al., 2015; Li et al., 2019; Moon & Kim, 2019; Yong, 2013) and references therein for examples of unconstrained optimal control laws in the continuous-time with finite and infinite horizon and with Markov jumps.

The reader is also referred to Ni, Zhang (2015), Yang et al. (2019) and Zhang et al. (2019) and references therein for the unconstrained optimal control laws of discrete-time infinite-horizon, and Benoussan et al. (2013), Huang and Li (2018), Gomes and Saude (2014), Carmona and Delarue (2018), Moon (2019) for applications on mean-field games.

Regarding the discrete-time mean-field finite-horizon problem, Elliott and Ni (2013), Ni, Elliott (2015), Ni et al. (2016b) and Ni et al. (2016a) investigated the unconstrained multi-period control problem of systems with multiplicative noises similar to ours. Elliott Ni (2013) provided a necessary and sufficient solvability condition alongside with an explicit optimal control using a matrix dynamical optimisation method. Ni et al. (2015) studied the case with indefinite cost weighting matrices and provided necessary and sufficient conditions for the solvability of the MV problem. It also showed a necessary condition for the solvability of the mean-field LQ problem based on the equivalence between the constrained generalised difference Riccati equations and a constrained linear recursive equation. Ni et al. (2016b) introduced a decomposition technique of the state and the control based on the modes of a Markov chain. Finally, Ni et al. (2016a) investigated the open and closed-loop optimal control with a more in-depth characterisation, difference and relationship between them.

As exemplified by the papers above, there is an extensive literature regarding the unconstrained multi-period optimal control strategy using the mean-field formulation. However, to the best of our knowledge, MV and constrained multi-period optimal control problems for systems with multiplicative noises lacks further investigation and poses new challenges in this field. The present paper aims to apply the mean-field formulation presented in Cui et al. (2014) to overcome the nonseparability difficulties associated with the MV control problems for the class of discrete-time linear systems with multiplicative noises. By doing this, we can handle the mathematical difficulties regarding the characterisation of optimal control laws and closed-form solutions for this class of MV control problems.

Our methodology consists of using the mean-field formulation to study a multi-period MV control problem for a discrete-time linear system subject to multiplicative noises, with state-variable \( x(k) \), by analysing its expected value, given by \( \bar{x}(k) \), together with \( z(k) = x(k) - \bar{x}(k) \). The optimal control law for this general multi-period MV problem is derived by using dynamic programming and, based on this solution, we obtain the optimal control strategy for the unconstrained problem. In order to solve the constrained problems, we adopt the Lagrangian multipliers approach to rewrite the problems with restrictions as unconstrained ones, and in one of these problems a closed-form solution is derived.

In contrast to previous papers, the main contributions of this paper are summarised as follows:

- We generalise the scalar unified framework in Cui et al. (2014) for discrete-time linear systems with multiplicative noises and obtain the multi-period optimal control law for a general MV problem. The optimal control strategy is derived from a set of two generalised Riccati difference equations and some parameters obtained from some recursive equations.
- Based on the solution to this general problem, we consider four other problems. The first one optimises an unconstrained trade-off between the expected output of the system and its variance. For the other three problems, we adopt the Lagrangian multipliers approach to rewrite the problems as unconstrained ones. The second one minimises the variance while keeping the expected output of the system constrained by a minimum value. The third performance criterion maximises the expected output of the system while keeping its variance constrained by a maximum value, and the fourth performance criterion maximises the expected output of the system while restricting its minimum value to a given probability of occurrence.
We derive a sufficient condition for a closed-solution for the problem of minimising the variance while keeping the expected output of the system constrained by a minimum value.

We show that when particularised to the portfolio optimisation problem, we retrieve the results obtained in Cui et al. (2014) using the mean-field formulation.

We present some numerical examples for the multi-period portfolio selection problem, in which it is desired to minimise the sum of the mean-variance trade-off costs of a portfolio against a benchmark along the time.

The paper is organised as follows. Section 2 shows the notation, some preliminary results, the problem definition and the mean-field formulation. In Section 3, we present some auxiliary results, and in Section 4, we obtain the main results of this paper. We illustrate our formulation to a portfolio allocation problem in Section 5, where we initially retrieve some known results in the literature, and later we present numerical examples for the multi-period mean-variance portfolio tracking problem. Section 6 concludes the paper with some final comments. In the Appendix, we present some auxiliary results.

2. Problem definition and mean-field formulation

2.1. Notation and auxiliary results

Throughout the paper, the $n$-dimensional real Euclidean space will be denoted by $\mathbb{R}^n$ and the normed bounded linear space of all $n \times m$ real matrices by $\mathbb{H}^{n,m}$ with $\mathbb{H}^n = \mathbb{H}^{n,n}$.

For $A \in \mathbb{H}^n$, we use the standard notation $A \geq 0$ ($A > 0$ respectively) to denote that the matrix $A$ is positive semi-definite (positive definite) and write $\mathbb{H}^{n,+}$ for the set of positive semi-definite matrices.

The range and the null spaces of a matrix $A \in \mathbb{H}^{n,m}$ will be denoted respectively by $\text{Im}(A)$ and $\text{Ker}(A)$, and $A'$ will represent the transpose of $A$.

We recall that $\text{Im}(A) = \text{Ker}(A')^\perp$, where $X^\perp$ represents the orthogonal complement of a linear subspace $X$.

For a matrix $A \in \mathbb{H}^{n,m}$, the generalised inverse of $A$ (or Moore-Penrose inverse of $A$) is defined to be the unique matrix $A^\dagger \in \mathbb{H}^{m,n}$ such that (i) $AA^\dagger A = A$, (ii) $A^\dagger A = A^\dagger$, (iii) $(AA^\dagger)' = AA^\dagger$ and (iv) $(A^\dagger A)' = A^\dagger A$ (see Saberi & Sannuti, 1995, pp. 12–13).

The operator expected value will be represented by $\mathbb{E}(\cdot)$. The following propositions will be useful in the sequel.

**Proposition 2.1:** Consider $Z \in \mathbb{H}^n$ and $M \in \mathbb{H}^{m}$ with $Z \geq 0$ and $M \succeq 0$. Let $A$ and $B$ be stochastic matrices (that is, each element of the matrix is a random variable) of appropriate dimensions.

Then $\mathbb{E}(A'ZA) - \mathbb{E}(A'ZB)(\mathbb{E}(B'ZB) + M)^\dagger \mathbb{E}(B'ZA) \geq 0$ and $\mathbb{E}(A'ZB) = \mathbb{E}(A'ZB)(\mathbb{E}(B'ZB) + M)^\dagger (\mathbb{E}(B'ZB) + M)$.

**Proof:** See Proposition 3 in Costa da Paul (2007).

**Proposition 2.2:** For $G = G' \in \mathbb{H}^n$ and $H \in \mathbb{H}^{n,m}$ it follows that $H(I - GG') = 0$ if and only if $\text{Ker}(G) \subseteq \text{Ker}(H)$.

**Proof:** See Lemma 4.2 in Rami et al. (2002).

2.2. Problem formulation

We consider the following linear system with multiplicative noise, on a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$, running up to a final time $T$:

$$x(k + 1) = \left( \bar{A}(k) + \sum_{s=1}^{\epsilon^x} \bar{A}_s(k)w_s^x(k) \right) x(k) + \left( \bar{B}(k) + \sum_{s=1}^{\epsilon^u} \bar{B}_s(k)w_s^u(k) \right) u(k),$$

$$x(0) = x_0, \quad k = 0, \ldots, T - 1. \quad (1)$$

We consider the following scalar output of system (1):

$$y(k) = L(k)x(k), \quad (2)$$

where $L(k) \in \mathbb{H}^{1,n}$.

Without loss of generality, we assume that $\epsilon = \epsilon^x = \epsilon^u$.

We have for each $k = 0, 1, \ldots, T - 1$, $\bar{A}(k) \in \mathbb{H}^n$, $\bar{A}_s(k) \in \mathbb{H}^n$, $s = 1, \ldots, \epsilon$, $\bar{B}(k) \in \mathbb{H}^{m,n}$, $\bar{B}_s(k) \in \mathbb{H}^{m,n}$, $s = 1, \ldots, \epsilon$.

The multiplicative noises \{ $w_s^x(k); s = 1, \ldots, \epsilon^x, k = 0, 1, \ldots, T - 1$ \} and \{ $w_s^u(k); s = 1, \ldots, \epsilon^u, k = 0, 1, \ldots,$ \}
are, without loss of generality, both zero-mean random variables with variance equal to 1, \( \mathbb{E}(w_i^x(k)w_j^y(k)) = 0 \) and \( \mathbb{E}(w_i^u(k)w_j^u(k)) = 0 \) for all \( k \) and \( i \neq j \).

We also assume that \( w_i^x(k), w_j^x(k'), w_i^u(k), w_j^u(k') \) are independent for \( k \neq k' \) and \( s, s' = 1, \ldots, \varepsilon \).

The mutual correlation between \( w_{s_1}^x(k) \) and \( w_{s_2}^u(k) \) is denoted by \( \rho_{s_1,s_2}(k) \).

The initial condition \( x_0 \) is assumed to be a vector in \( \mathbb{R}^n \).

We define \( \mathcal{F}_t \) as the \( \sigma \)-field generated by \( \{w_i^x(k), w_i^u(k); s = 1, \ldots, \varepsilon, k = 0, \ldots, \tau - 1\} \) for \( \tau = 1, \ldots, T \), and \( \mathcal{F}_0 \) the trivial \( \sigma \)-field over \( \Omega \), so that the expected value \( \mathbb{E}(\cdot | \mathcal{F}_0) \) is just the unconditional expected value \( \mathbb{E}(\cdot) \).

We write \( \mathcal{Q}(k) = \{u(k); u(k) \text{ is an } m\text{-dimensional random vector with finite second moments and } \mathcal{F}_k\text{-measurable}\} \) and \( \mathcal{Q}(\tau) = \{u_\tau = (u(\tau), \ldots, u(T - 1)); u(k) \in \mathcal{Q}(k) \text{ for each } k = \tau, \ldots, T - 1\} \).

For simplicity we write \( \mathcal{U} = \mathcal{U}(0) \).

The superscript \( u \) will indicate that the control law \( u \) is being applied to Equations (1) and (2).

The mean-variance general problem, denoted by \( PG(v, \xi, l, D) \), will be used as a base problem to solve all four specific problems and is defined as:

\[
PG(v, \xi, l, D) : \min_{u \in \mathcal{U}} \sum_{t=1}^{T} \left( v(t) \text{Var} (y^u(t)) - \xi(t)\mathbb{E}(y^u(t)) - l(t)\mathbb{E}(y^u(t))^2 + D(t) \right),
\]

where the vectors \( v' = [v(1), \ldots, v(T)] \geq 0 \) and \( \xi' = [\xi(1), \ldots, \xi(T)] \geq 0 \) are the input parameters for this problem. They can be seen as risk aversion coefficients, giving a trade-off preference between the expected output and the associated risk (variance) level at time \( t \).

We also have the input parameter \( l' = [l(1), \ldots, l(T)] \), associated with the squared expected output and \( D' = [D(1), \ldots, D(T)] \) introduced just to help the notation of problems to be defined later. Since \( D \) does not depend on the control variable, it could be removed from the optimisation problem.

The parameters \( l(t) \) and \( D(t) \) will be appropriately specified in the sequel.

**Remark 2.1:** In what follows it will be convenient to set \( v(0) = 0, \xi(0) = 0, l(0) = 0 \) and \( D(0) = 0 \).

The mean-variance unconstrained problem is defined as:

\[
PU(v, \xi) : \min_{u \in \mathcal{U}} \sum_{t=1}^{T} \left( v(t) \text{Var} (y^u(t)) - \xi(t)\mathbb{E}(y^u(t)) \right),
\]

and in this case, we wish an optimal control with no restriction in neither the expected output nor its variance.

Notice that Problem \( PU(v, \xi) \) in Equation (4) can be rewritten as in Equation (3) by taking \( l(t) = 0 \) and \( D(t) = 0, t = 1, \ldots, T \).

Providing an analytical solution to the optimal control law that takes into consideration a restriction on either the minimum expected output or the maximum variance over time would be relevant to extend the applicability of our formulation.

Portfolio managers of pension funds, for instance, would be interested in achieving a return above inflation or even defining a portfolio that has limited risk over specific periods.

These two constrained problems are defined as:

\[
PC1(v, \xi) : \min_{u \in \mathcal{U}} \sum_{t=1}^{T} \left( v(t) \text{Var} (y^u(t)) \right)
\]

s.t. : \( \mathbb{E}(y^u(t)) \geq \xi(t) \) \quad (5)

and

\[
PC2(\xi, \varphi) : \min_{u \in \mathcal{U}} \sum_{t=1}^{T} \left( \xi(t)\mathbb{E}(y^u(t)) \right)
\]

s.t. : \( \text{Var}(y^u(t)) \leq \varphi(t) \), \quad (6)

for \( t = 1, \ldots, T \).

In problem \( PC1 \), we wish the control strategy that minimises the weighted sum of the variance while restricting the expected return to a minimum value, \( \xi(t) \).

In problem \( PC2 \), we wish the control strategy that maximises the weighted sum of the expected output while restricting its variance to a maximum value, \( \varphi(t) \).

As before, \( v \) and \( \xi \) are input parameters that represent a trade-off between risk and return over time.

Another relevant problem for investors, for instance, would involve the risk control that maintains the portfolio value above a minimum value with a given probability.
This dynamic mean-variance problem with risk control over a minimum expected output subjected to a maximum probability of occurrence is formulated as the following problem,

$$\min_{u \in U} - \sum_{t=1}^{T} \left( \xi(t) \mathbb{E} \left( y^u(t) \right) \right)$$

s.t. : $P(x(t) \leq b(t)) \leq a(t)$, \hspace{1cm} (7)

for $t = 1, \ldots, T$, where $b(t)$ is the disaster level of the output and $a(t)$ is its acceptable maximum probability.

In a portfolio management perspective, for instance, $b(t)$ can be considered as the minimum level of capital and $a(t)$ as the maximum acceptable probability of achieving $b(t)$.

As the problem defined in Equation (7) is hard to be directly solved, we replace $P(x(t) \leq b(t))$ by its upper bound $\text{Var}(y^u(t))/[\mathbb{E}(y^u(t)) - b(t)]^2$ using Tchebycheff inequality as proposed in Zhu et al. (2004), resulting in the following generalised mean-variance model,

$$PC3 \ (\xi, a, b) : \min_{u \in U} - \sum_{t=1}^{T} \left( \xi(t) \mathbb{E} \left( y^u(t) \right) \right)$$

s.t. : $\text{Var} \left( y^u(t) \right)$

$$\leq a(t) \left[ \mathbb{E} \left( y^u(t) \right) - b(t) \right]^2,$$ \hspace{1cm} (8)

for $t = 1, \ldots, T$.

The optimal solution to Problem $PC3(\xi, a, b)$ is feasible to Problem (7), thus serving as an approximate solution for this problem.

### 2.3. Lagrangian optimisation problems

In order to solve Problems (5), (6), and (8), we adopt, as in Zhu et al. (2004), a primal-dual method by attaching the constraints to the objective function through the Lagrangian multipliers $\omega = [\omega(1), \ldots, \omega(T)]$, $\omega(t) \geq 0$, $t = 1, \ldots, T$. The new problems take the following unconstrained forms:

$$PL1(\omega) : \min_{u \in U} \sum_{t=1}^{T} \left( v(t) \text{Var} \left( y^u(t) \right) + \omega(t) \left( \epsilon(t) - \mathbb{E} \left( y^u(t) \right) \right) \right),$$ \hspace{1cm} (9)

$$PL2(\omega) : \min_{u \in U} \sum_{t=1}^{T} \left( \omega(t) \left( \text{Var} \left( y^u(t) \right) - \varphi(t) \right) \right)$$

### Table 1. Input parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PU</th>
<th>PL1</th>
<th>PL2</th>
<th>PL3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(k)$</td>
<td>$v(k)$</td>
<td>$v(k)$</td>
<td>$\omega(k)$</td>
<td>$\omega(k)$</td>
</tr>
<tr>
<td>$\xi(k)$</td>
<td>$\xi(k)$</td>
<td>$\omega(k)$</td>
<td>$\xi(k)$</td>
<td>$\xi(k)$</td>
</tr>
<tr>
<td>$l(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\omega(k)a(k)b(k)$</td>
</tr>
<tr>
<td>$D(k)$</td>
<td>0</td>
<td>$\omega(k)\epsilon(k)$</td>
<td>$-\omega(k)\xi(k)$</td>
<td>$-\omega(k)a(k)b(k)$</td>
</tr>
</tbody>
</table>

Table 1.

In all the above problems the next step it to solve the Lagrangian dual problem $PCi = \max_{\omega \geq 0} \mathcal{H}(\omega)$ where $\mathcal{H}(\omega) = PLi(\omega)$, $i = 1, 2$ or 3 (see also Bazarra et al., 2013). Notice that Equations (4), (9), (10) and (11) can be rewritten as in Equation (3) by choosing the parameters $v(k)$, $\xi(k)$, $l(k)$ and $D(k)$ as in Table 1.

### 2.4. Mean-field formulation

We consider now the mean-field formulation for the general problem $PG$.

For that we define $\tilde{x}(k) = \mathbb{E}(x(k))$, $z(k) = x(k) - \tilde{x}(k)$, $\bar{u}(k) = \mathbb{E}(u(k))$, $v(k) = u(k) - \bar{u}(k)$.

From Equation (1) and the independence hypothesis made on the multiplicative noises, we get that

$$\tilde{x}(k + 1) = \tilde{A}(k)\tilde{x}(k) + \tilde{B}(k)\bar{u}(k),$$

$$\tilde{x}(0) = x_0, \hspace{1cm} k = 0, \ldots, T - 1,$$ \hspace{1cm} (12)

and

$$z(k + 1) = \left( \tilde{A}(k) + \sum_{s=1}^{g^x} \tilde{A}_s(k)w_s^x(k) \right) z(k)$$

$$+ \sum_{s=1}^{g^x} \tilde{A}_s(k)w_s^x(k)\tilde{x}(k)$$

$$+ \left( \tilde{B}(k) + \sum_{s=1}^{g^y} \tilde{B}_s(k)w_s^y(k) \right) v(k)$$
We define $S(k)$, $\mathbb{V}(\tau)$, $\mathbb{M}(\tau)$ as follows: we say that $(\bar{u}(k), v(k)) \in S(k)$ if $\bar{u}(k) \in \mathbb{R}^m$ and $v(k) \in Q(k)$ satisfying $E(v(k)) = 0$, that $(\bar{u}_r, \nu_r) \in \mathbb{V}(\tau)$ if $(\bar{u}_r, \nu_r) = ((\bar{u}(\tau), v(\tau)), \ldots, (\bar{u}(T-1), v(T-1)))$ with $(\bar{u}(k), v(k)) \in S(k)$ for each $k = r, \ldots, T-1$, and that $(\bar{u}^*, v^*) = ((\bar{u}(0), v(0)), \ldots, (\bar{u}(\tau), v(\tau))) \in \mathbb{M}(\tau)$ if $(\bar{u}(k), v(k)) \in S(k)$ for each $k = 0, \ldots, \tau$. As before we set $\mathbb{V} = \mathbb{V}(0)$ and write $(\bar{u}, v) = (\bar{u}_0, v_0) \in \mathbb{V}$. Problem $PG(u, \xi, l, D)$ in Equation (3) can be rewritten now as

$$
PG (v, \xi, l, D) : J_0 (\bar{x}(0), z(0))
$$

$$
= \min_{(\bar{u}, v) \in \mathbb{V}} \sum_{t=1}^{T} \mathbb{E} \left( v(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - l(t)(L(t)\bar{x}(t))^2 + D(t) \right),
$$

(14)

with $\bar{x}$ and $z$ satisfying Equations (12) and (13). By the fact that $E(v(k)) = 0$, we get from Equation (13) that $E(z(k)) = 0$ for all $k = 0, \ldots, T$. Indeed, by induction, clearly we have that $E(z(0)) = 0$ and, considering $E(z(k)) = 0$, we have from the independence hypothesis made on the multiplicative noises and Equation (13) that

$$
E(z(k + 1))
$$

$$
= \left( \tilde{A}(k) + \sum_{s=1}^{g^x} \widetilde{A}_s(k) E(w^x_s(k)) \right) E(z(k))
$$

$$
+ \sum_{s=1}^{g^x} \widetilde{A}_s(k) E(w^x_s(k)) \bar{x}(k)
$$

$$
+ \left( \tilde{B}(k) + \sum_{s=1}^{g^u} \widetilde{B}_s(k) E(w^u_s(k)) \right) E(v(k))
$$

$$
+ \sum_{s=1}^{g^u} \widetilde{B}_s(k) E(w^u_s(k)) \bar{u}(k) = 0.
$$

At each time $k \in \{1, \ldots, T\}$ and for any $(\bar{u}^{k-1}, v^{k-1}) \in \mathbb{M}(k - 1)$, define the following intermediate problem for Problem (14):

$$
J_k \left( \bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1}) \right)
$$

$$
= \min_{(\bar{u}, v) \in \mathbb{V}(k)} \sum_{t=k}^{T} \mathbb{E} \left( v(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - l(t)(L(t)\bar{x}(t))^2 + D(t) \mid \mathcal{F}_k \right).
$$

(15)

We have the following result.

**Lemma 2.1:** Assume that for $t \in \{1, \ldots, T\}$ and any $(\bar{u}^{t-1}, v^{t-1}) \in \mathbb{M}(t - 1)$,

$$
\mathbb{E}(f_t(\bar{x}(t), z(t), (\bar{u}^{t-1}, v^{t-1}) \mid \mathcal{F}_{t-1})
$$

$$
= G_{t-1}^1(\bar{x}(t-1), z(t-1), (\bar{u}^{t-1}, v^{t-1}))
$$

$$
+ G_{t-1}^2(\bar{x}(t-1), z(t-1), (\bar{u}^{t-1}, v^{t-1}))
$$

(16)

with

$$
\mathbb{E}(G_{t-1}^2(\bar{x}(t-1), z(t-1), (\bar{u}^{t-1}, v^{t-1}))) = 0.
$$

(17)

Then for $t = 0, \ldots, T - 1$,

$$
(\bar{u}^*(t), v^*(t)) \in \arg \min_{(\bar{u}(t), v(t)) \in \mathbb{S}(t)} \left\{ G_t^1(\bar{x}(t), z(t), (\bar{u}^{t-1}, v^{t-1}))
$$

$$
+ v(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - l(t)(L(t)\bar{x}(t))^2 + D(t) \right\}
$$

and

$$
J_0 (\bar{x}(0), z(0))
$$

$$
= \min_{(\bar{u}, v) \in \mathbb{M}(t)} \left\{ \mathbb{E} \left( G_t^1(\bar{x}(t), z(t), (\bar{u}^{t-1}, v^{t-1})) \right)
$$

$$
+ \sum_{j=0}^{t} \mathbb{E} \left( v(j)(L(j)z(j))^2 - \xi(j)L(j)\bar{x}(j) - l(j)(L(j)\bar{x}(j))^2 + D(j) \right) \right\},
$$

i.e. $G_t^1(\bar{x}(t), z(t), ((\bar{u}^{t-1}, v^{t-1}), (\bar{u}(t), v(t))) + v(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - l(t)(L(t)\bar{x}(t))^2 + D(t)$ can be regarded as the benefit-to-go function at time $t$ of problem $PG$.

**Proof:** It is an immediate application of Lemma 3 in Cui et al. (2014)
Remark 2.2: Notice that we can rewrite Equation (9) using the mean-field formulation as

\[ PL1(\omega) : \int_0^{PL1} (\tilde{x}(0), z(0)) \]

\[ = \min_{(\tilde{u}, \tilde{v}) \in \T} \sum_{t=1}^T \mathbb{E} \left( v(t)(L(t)z(t))^2 - \omega(t)L(t)\tilde{x}(t) + \omega(t)\psi(t) \right) \]  

(18)

Similarly, we can rewrite Equation (10) using the mean-field formulation as

\[ PL2(\omega) : \int_0^{PL2} (\tilde{x}(0), z(0)) \]

\[ = \min_{(\tilde{u}, \tilde{v}) \in \T} \sum_{t=1}^T \mathbb{E} \left( \omega(t)(L(t)z(t))^2 - \xi(t)L(t)\tilde{x}(t) - \zeta(t)\tilde{x}(t) - \omega(t)a(t)b(t)\right) \]  

(19)

and Equation (11) as

\[ PL3(\omega) : \int_0^{PL3} (\tilde{x}(0), z(0)) \]

\[ = \min_{(\tilde{u}, \tilde{v}) \in \T} \sum_{t=1}^T \mathbb{E} \left( \omega(t)(L(t)z(t))^2 - \xi(t)L(t)\tilde{x}(t) - \omega(t)a(t)b(t)\right) \]  

(20)

where for \( t = 1, \ldots, T \),

\[ \xi(t)(t) = \xi(t)L(t) - 2\omega(t)a(t)b(t) \]  

(21)

Finally, Problem \( PU(v, \xi) \) in Equation (4) can be rewritten using the mean-field formulation as

\[ PU(v, \xi) := \min_{(\tilde{u}, \tilde{v}) \in \T} \sum_{t=1}^T \mathbb{E} \left( v(t)(L(t)z(t))^2 - \xi(t)L(t)\tilde{x}(t) \right) \]  

(22)

3. Main operators and auxiliary results

For \( k = 0, \ldots, T - 1 \), the following operators \( \mathcal{A}(k, \ldots) \in \mathbb{H}^n \), \( \mathcal{G}(k, \ldots) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^m \), \( \mathcal{R}(k, \ldots) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^m \), and the nonlinear operators \( \mathcal{K}(k, \ldots), \mathcal{M}(k, \ldots) \) and \( \mathcal{P}(k, \ldots) \) will be useful in the sequel.

For \( X, Y \in \mathbb{H}^n \),

\[ \mathcal{A}(k, X, Y) = \tilde{A}(k)X\tilde{A}(k) + \sum_{s=1}^k \tilde{A}_s(k)Y\tilde{A}_s(k), \]  

(23)

Define also the nonlinear operators \( \mathcal{V}(k, \ldots) \) and \( \mathcal{D}(k, \ldots) \) as follows. For \( X, Y \in \mathbb{H}^n, V \in \mathbb{H}^n, \gamma \in \mathbb{R}, \)

\[ \mathcal{V}(k, X, Y, V) = V \left( \tilde{A}(k) - \tilde{B}(k) \right) \mathcal{K}(k, X, Y) \]

(24)

\[ + \xi(k)L(k) \] and

\[ \mathcal{D}(k, X, Y, \gamma) = \gamma - \frac{1}{4}V\tilde{B}(k)\mathcal{R}(k, X, Y)^\dagger \]

\[ \times \tilde{B}(k)\gamma' + D(k). \]  

(25)

The following auxiliary proposition will be needed in the sequel.

Proposition 3.1: For \( X, Y \in \mathbb{H}^n \), we have that \( \mathcal{P}(k, X) \in \mathbb{H}^n, \mathcal{M}(k, X, Y) \in \mathbb{H}^n \) and

\[ \mathcal{G}(k, X, Y)' = \mathcal{G}(k, X, Y)\mathcal{R}(k, X, Y)^\dagger \mathcal{R}(k, X, Y). \]  

(26)

Proof: Set in Proposition 2.1 \( M = 0, \)

\[ A = \begin{bmatrix} \tilde{A}(k) \\ \sum_{s=1}^k \tilde{A}(s)w_s^2(k) \end{bmatrix}, \]

\[ B = \begin{bmatrix} \tilde{B}(k) \\ \sum_{s=1}^k \tilde{B}(s)w_s^2(k) \end{bmatrix}, \] and

\[ Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq 0. \]  

(27)

Then, from the hypothesis made for \( \{w_s^2(k)\} \) and \( \{w_s^2(k)\} \), we have that \( \mathbb{E}(A'ZA) = \mathcal{A}(k, X, Y), \mathbb{E}(A' \mathcal{P}(k, X) \mathcal{P}(k, X)^\dagger A) = \mathcal{M}(k, X, Y)^2 \mathcal{R}(k, X, Y)^\dagger \mathcal{R}(k, X, Y). \)
Proposition 3.2: From Proposition 3.1, we have the following result.

\[ H = \mathcal{M}(k, X, Y) = \mathbb{E}(A'ZA) - \mathbb{E}(A'ZB)(\mathbb{E}(B'ZB)) \]

Then Equation (26) follows from Proposition 2.1. \[ \blacksquare \]

We will use the Bellman optimality equation, written in terms of the operators above, to solve Problem (14).

Define the following sequences for \( k = T, T-1, \ldots, 0, \)

\[
P(k) = P(k, P(k + 1)),
\]

\[
P(T) = v(T)L(T)'L(T),
\]

\[
M(k) = \tilde{M}(k, M(k + 1), P(k + 1)),
\]

\[
M(T) = -I(T)L(T)'L(T),
\]

\[
V(k) = V(k, M(k + 1), P(k + 1), V(k + 1)),
\]

\[
V(T) = \xi(T)L(T), \quad \text{and}
\]

\[
\gamma(k) = \mathcal{D}(k, M(k + 1), P(k + 1),
\]

\[
V(k + 1), \gamma(k + 1)), \quad \gamma(T) = D(T).
\]

\[ (31) \]

**Remark 3.1:** If \( \tilde{A}_s(k) = 0, k = 0, \ldots, T-1 \) and \( l(k) = 0, k = 1, \ldots, T, \) then \( M(k) = 0 \) for all \( k = 0, \ldots, T. \) Indeed, by applying induction on \( k = T, T-1, \ldots, 0 \) in Equation (29) we have, by definition, that \( M(T) = 0. \) Now supposing that \( M(k + 1) = 0 \) we get that \( A(k, M(k + 1), P(k + 1)) = A(k, 0, P(k + 1)) = 0 \) and \( \mathcal{G}(k, M(k + 1), P(k + 1)) = \mathcal{G}(k, 0, P(k + 1)) = 0 \) since \( \tilde{A}_s(k) = 0, \) and thus \( M(k) = \mathcal{M}(k, M(k + 1), P(k + 1)) = \mathcal{M}(k, 0, P(k + 1)) = 0 \) (notice that in this case, \( \tilde{M} = \mathcal{M}, \) completing the induction arguments.

Set also

\[
K(k) = \mathcal{R}(k, P(k + 1), P(k + 1))
\]

\[ (32) \]

\[
H(k) = \mathcal{R}(k, M(k + 1), P(k + 1))
\]

\[ (33) \]

From Proposition 3.1, we have the following result.

**Proposition 3.2:** We have that \( P(k) \in \mathbb{H}^{n+}, M(k) \in \mathbb{H}^{n+}, \)

\[
\mathcal{G}(k, P(k), P(k))' = \mathcal{G}(k, P(k), P(k))'\mathcal{R}(k, P(k),
\]

\[
\mathcal{R}(k, P(k), P(k)), \quad \text{and}
\]

\[
\mathcal{G}(k, M(k), P(k))' = \mathcal{G}(k, M(k), P(k))'\mathcal{R}(k, M(k),
\]

\[
P(k))'\mathcal{R}(k, M(k), P(k)). \quad (35)
\]

**Proof:** The result follows from Proposition 3.1, after induction on \( k \) to show that \( P(k) \in \mathbb{H}^{n+}, M(k) \in \mathbb{H}^{n+}, \) for all \( k = T, \ldots, 0. \) \[ \blacksquare \]

We make the following assumption:

**Assumption 3.1:** We assume that for \( k = 0, \ldots, T - 1, \)

\[
\tilde{B}(k)'V(k + 1) = \text{Im}(\mathcal{R}(k, M(k + 1), P(k + 1))) \quad (36)
\]

\[
\mathcal{R}(k, M(k + 1), P(k + 1)) \geq 0. \quad (37)
\]

**Remark 3.2:** Notice that from Proposition 3.1 and Equation (28), we have that \( P(k) \geq 0 \) for all \( k = 0, \ldots, T \) since \( P(T) \geq 0. \) If \( l(k) = 0 \) for all \( k = 1, \ldots, T \) (as in Problems PU, PL1 and PL2) then from Equation (23), we get that \( \mathcal{M} = \mathcal{M} \) so that from Proposition 3.1 and Equation (29) we have that \( M(t) \geq 0 \) for all \( t = 0, \ldots, T. \)

In this case, \( l(k) = 0, k = 1, \ldots, T, \) and we only require Equation (36) in Assumption 3.1, since from the definition of \( \mathcal{R} \) in Equation (23) and the fact that \( M(k) \geq 0, P(k) \geq 0 \) for all \( k = 0, \ldots, T, \) we get that Equation (37) will always be satisfied.

We have the following result:

**Proposition 3.3:** We have that for \( k = 0, \ldots, T - 1, \)

\[
V(k + 1)\tilde{B}(k)' = V(k + 1)\tilde{B}(k)'\mathcal{R}(k, M(k + 1),
\]

\[
P(k + 1))'\mathcal{R}(k, M(k + 1), P(k + 1)). \quad (38)
\]

**Proof:** Set for simplicity \( R = \mathcal{R}(k, M(k + 1), P(k + 1)) \) and \( H = \tilde{B}(k)'V(k + 1)' \). Since \( \text{Im}(R) = \text{Im}(R^\dagger) \) and \( \text{Im}(R^\dagger) = \text{Ker}(R^\dagger) \), we have from Equation (36) that \( H \in \text{Im}(R^\dagger), \) and thus \( \text{Ker}(R^\dagger) \subseteq \text{Ker}(H^\dagger). \)

From Proposition 2.2, we get Equation (38). \[ \blacksquare \]
Conditions (36) and (37) are equivalent to the following computationally easier to check condition:

\[
\begin{bmatrix}
\dot{B}(k)V(k+1) & \mathcal{R}(k,M(k+1),P(k+1))
\end{bmatrix}
\begin{bmatrix}
\ddot{B}(k)V(k+1) \\
\mathcal{R}(k,M(k+1),P(k+1))
\end{bmatrix}
\geq 0.
\] (39)

Indeed, from Schur’s complement, (39) is equivalent to \( \mathcal{R}(k,M(k+1),P(k+1)) \geq 0 \) and (38) (see Saberi Sannuti (1995), pp. 12–13), which is equivalent to (36), see Proposition 3.3.

4. Main results

Define for \((\ddot{u}^k, v^k) \in \mathbb{M}(k), k = 0, \ldots, T-1,\)

\[
G_k^1(\ddot{x}(k), z(k), (\ddot{u}^k, v^k)) = z(k)'(\mathcal{A}(k,P(k+1),P(k+1))z(k)
+ \ddot{x}(k)'(\mathcal{A}(k,M(k+1),P(k+1)))\ddot{x}(k)
+ v(k)'(\mathcal{R}(k,P(k+1),P(k+1)))v(k)
+ \ddot{u}(k)'(\mathcal{R}(k,M(k+1),P(k+1)))\ddot{u}(k)
+ 2z(k)G(k,P(k+1),P(k+1))'v(k)
+ 2\ddot{x}(k)G(k,M(k+1),P(k+1))'\ddot{u}(k)
- V(k+1)(\ddot{A}(k)\ddot{x}(k) + \ddot{B}(k)\ddot{u}(k)) + \gamma(k+1)
\] (40)

and

\[
G_k^2(\dddot{x}(k), z(k), (\ddot{u}^k, v^k))
= 2z(k)'
\left(\sum_{\sigma=1}^{\ell} \ddot{A}_\sigma(k)P(k)\dddot{A}_\sigma(k)\ddot{x}(k)
+ \sum_{\sigma_1=1}^{\ell} \sum_{\sigma_2=1}^{\ell} \rho_{\sigma_1,\sigma_2}(k)\ddot{A}_{\sigma_1}(k)P(k)\dddot{B}_{\sigma_2}(k)
\right)\ddot{u}(k)
+ 2\dddot{x}(k)'
\left(\sum_{\sigma_1=1}^{\ell} \sum_{\sigma_2=1}^{\ell} \rho_{\sigma_1,\sigma_2}(k)\ddot{A}_{\sigma_1}(k)P(k)\dddot{B}_{\sigma_2}(k)
\right)\ddot{u}(k)
+ \dddot{u}(k)'
\left(\sum_{\sigma_1=1}^{\ell} \dddot{B}_{\sigma_1}(k)P(k)\dddot{B}_{\sigma_1}(k)
\right)v(k).
\] (41)

Note that

\[
\mathbb{E}(G_k^2(\dddot{x}(k), z(k), (\ddot{u}^k, v^k)))
\]

\[
= 2 \left( \mathbb{E}(z(k))' \left( \sum_{\sigma=1}^{\ell} \dddot{A}_\sigma(k)P(k)\dddot{A}_\sigma(k) \right) \ddot{x}(k)
+ \left( \sum_{\sigma_1=1}^{\ell} \sum_{\sigma_2=1}^{\ell} \rho_{\sigma_1,\sigma_2}(k)\dddot{A}_{\sigma_1}(k)P(k)\dddot{B}_{\sigma_2}(k) \right)\ddot{u}(k)
+ 2 \left( \dddot{x}(k)' \sum_{\sigma_1=1}^{\ell} \sum_{\sigma_2=1}^{\ell} \rho_{\sigma_1,\sigma_2}(k)\dddot{A}_{\sigma_1}(k)P(k)\dddot{B}_{\sigma_2}(k) \right)\ddot{u}(k)
\right)\mathbb{E}(v(k)) = 0.
\]

since \(\mathbb{E}(z(k)) = 0\) and \(\mathbb{E}(v(k)) = 0\).

**Theorem 4.1:** Suppose that Assumption 3.1 holds. We have that

\[
J_k \left( \ddot{x}(k), z(k), (\ddot{u}^{k-1}, v^{k-1}) \right)
= z(k)'P(k)z(k) + \ddot{x}(k)'M(k)\ddot{x}(k)
- V(k)\ddot{x}(k) + \gamma(k)
\] (42)

and Equation (16) is satisfied with \(G_k^1\) and \(G_k^2\) as in Equations (40) and (41), respectively. Moreover, the optimal control strategy for Problem (3) is given by \(u^*(k) = v^*(k) + \ddot{u}^*(k)\), where

\[
v^*(k) = -K(k)z(k) \quad \text{and} \quad (43)
\]

\[
\ddot{u}^*(k) = -H(k)\ddot{x}(k) + \frac{1}{2}\mathcal{R}(k,M(k+1),P(k+1))' \times \ddot{B}(k)'V(k+1)'.
\] (44)

**Proof:** We apply backward induction on \(k\). For \(k = T\), we have that

\[
J_T \left( \dddot{x}(T), z(T), (\dddot{u}^{T-1}, v^{T-1}) \right)
= v(T)z(T)'L(T)z(T)
- l(T)\dddot{x}(T)L(T)\dddot{L}(T)\dddot{x}(T)'
- \xi(T)L(T)\dddot{x}(T) + D(T)
\]

and the results follow with \(P(T) = v(T)L(T)'L(T)\), \(M(T) = -l(T)L(T)'L(T)\), \(V(T) = \xi(T)L(T)\) and \(\gamma(T) = D(T)\). Suppose that Equation (42) holds for
Let us evaluate each term in Equation (45). For the first term, we have from Equation (13) that

$$
\mathbb{E}(z(k + 1)P(k + 1)z(k + 1) | \mathcal{F}_k) = z(k)^{\prime}(\bar{A}(k)^{\prime} P(k + 1)\bar{A}(k) + \sum_{s = 1}^{g} \bar{A}_s(k)^{\prime} P(k + 1)\bar{A}_s(k)) z(k)
$$

$$
\quad + \bar{x}(k)^{\prime} \left( \sum_{s = 1}^{g} \widetilde{A}_s(k)^{\prime} P(k + 1)\bar{A}_s(k) \right) \bar{x}(k)
$$

$$
\quad + v(k)^{\prime} \left( \tilde{B}(k)^{\prime} P(k + 1)\tilde{B}(k) \right) v(k)
$$

For the second term, we have from Equation (12) that

$$
\mathbb{E}(\bar{x}(k + 1)^{\prime} M(k + 1)\bar{x}(k + 1) | \mathcal{F}_k) = \bar{x}(k + 1)^{\prime} M(k + 1)\bar{x}(k + 1)
$$

$$
\quad = \bar{x}(k)^{\prime} \left( A(k)^{\prime} M(k + 1)A(k) \right) \bar{x}(k)
$$

$$
\quad + 2\bar{x}(k)^{\prime} \left( \sum_{s = 1}^{g} \bar{A}_s(k)^{\prime} P(k + 1)\bar{A}_s(k) \right) \bar{x}(k)
$$

For the third term, we have again from Equation (12) that

$$
\mathbb{E}(V(k + 1)\bar{x}(k + 1) | \mathcal{F}_k) = V(k + 1)\bar{x}(k + 1)
$$

$$
\quad = V(k + 1)A(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k).
$$

For the fourth term, we get from Equation (45) that Equations (16) and (17) are satisfied with $G^1_k$ and $G^2_k$ as respectively in Equations (40) and (41).

Notice now that we can write the benefit-to-go function at time $k$ using Lemma 2.1 as

$$
G^1_k(\bar{x}(k), z(k), ((\bar{u}^{k-1}, v^{k-1}), (\bar{u}^{k}, v^{k})))
$$

$$
\quad + v(k)(L(k)z(k))^2 - l(k)(L(k)\bar{x}(k))^2
$$

$$
\quad - \xi(k)L(k)\bar{x}(k) + D(k)
$$

$$
= z(k)^{\prime}(A(k, P(k + 1), P(k + 1))
$$

$$
\quad + v(k)L(k)^{\prime} \bar{B}(k)z(k)
$$

$$
\quad + 2\bar{x}(k)^{\prime} \left( \sum_{s = 1}^{g} \bar{A}_s(k)^{\prime} P(k + 1)\bar{A}_s(k) \right) \bar{x}(k)
$$

$$
\times P(k + 1)\tilde{B}_s(k) v(k)
$$

$$
\quad + 2\tilde{x}(k)^{\prime} \left( \sum_{s = 1}^{g} \rho_{s_1, s_2}(k)\bar{A}_s(k)^{\prime} P(k + 1)\bar{A}_s(k) \right) \tilde{x}(k)
$$

$$
\times P(k + 1)\tilde{B}_s(k) v(k)
$$

$$
\quad + F_1(z(k), v(k), k) + F_2(\tilde{x}(k), \bar{u}(k), k).
$$

(49)
where

\[
F_1(z(k), v(k), k) = v(k)' \mathcal{R}(k, P(k+1), P(k+1)) v(k) + 2z(k)' \mathcal{G}(k, P(k+1), P(k+1))' v(k)
\]

and

\[
F_2(\tilde{x}(k), \tilde{u}(k), k) = \tilde{u}(k)' \mathcal{R}(k, M(k+1), P(k+1)) \tilde{u}(k) + \left[2\tilde{x}(k)' \mathcal{G}(k, M(k+1), P(k+1))' \tilde{u}(k) - V(k+1) \tilde{B}(k) \right] \tilde{u}(k).
\]

For simplicity we set next \( \mathcal{R}_1 = \mathcal{R}(k, P(k+1), P(k+1)), \mathcal{R}_2 = \mathcal{R}(k, M(k+1), P(k+1)), \mathcal{G}_1 = \mathcal{G}(k, P(k+1), P(k+1)) \) and \( \mathcal{G}_2 = \mathcal{G}(k, M(k+1), P(k+1)). \)

From Proposition 3.2 and the properties of the generalised inverse, it follows that

\[
F_1(z(k), v(k), k) = (v(k) + K(k)z(k))' \mathcal{R}_1(v(k) + K(k)z(k)) - z(k)' \mathcal{G}'_1 \mathcal{R}^+_1 \mathcal{G}_1 z(k)
\]

and from Proposition 3.3,

\[
F_2(\tilde{x}(k), \tilde{u}(k), k) = \left( \tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right) \right) \times \mathcal{R}_2 \left( \tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right) \right).
\]

Replacing Equations (50) and (51) into Equation (49), we get that

\[
\begin{align*}
G_k^1(\tilde{x}(k), z(k), ((\tilde{u}^{k-1}, v^{k-1}), (\tilde{u}^k, v^k))) & = z(k)' (P(k, P(k+1))) z(k) + \tilde{x}(k)' \\
& \quad \times (M(k, M(k+1), P(k+1)) \\
& \quad - l(k)(L(k)\tilde{x}(k)) - D(k) \\
& \quad - l(k)(L(k)\tilde{x}(k)) - (V(k+1)(A(k) \\
& \quad - \tilde{B}(k)H(k)) + \xi(k)(L(k)\tilde{x}(k) \\
& - \frac{1}{4} V(k+1)\tilde{B}(k) \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \\
& + \gamma(k+1) + D(k)) + (v(k) \\
& + K(k)z(k))' \mathcal{R}_1(v(k) + K(k)z(k)) + (\tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) \\
& - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right)) \times \mathcal{R}_2 \left( \tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right) \right).
\end{align*}
\]

We get that minimising the left hand side of Equation (52) in \( v(k) \) and \( \tilde{u}(k) \) is equivalent to minimise \( \phi_1(v(k)) \) and \( \phi_2(\tilde{u}(k)) \) since the other terms do not depend on \( v(k) \) and \( \tilde{u}(k) \). Since \( \mathcal{R}_1 \geq 0 \) and \( \mathcal{R}_2 \geq 0 \), the minimum is \( \phi_1(v^*(k)) = 0 \) and \( \phi_2(\tilde{u}^*(k)) = 0 \), with \( v^*(k) \) and \( \tilde{u}^*(k) \) given as in Equations (43) and (44). Note that \( \mathbb{E}(v^*(k)) = -K(k)\mathbb{E}(z(k)) = 0 \) and thus \( (\tilde{u}^*(k), v^*(k)) \in \mathbb{S}(k) \). Since

\[
J_k \left( \tilde{x}(k), z(k), ((\tilde{u}^{k-1}, v^{k-1}), (\tilde{u}^k, v^k)) \right) = \min_{(\tilde{u}(k), v(k)) \in \mathbb{S}(k)} \{ G_k^1(\tilde{x}(k), z(k), ((\tilde{u}^{k-1}, v^{k-1}), (\tilde{u}^k, v(k))) + v(k)(L(k)z(k))^2 \\
\quad - l(k)(L(k)\tilde{x}(k)) - D(k) \\
\quad - l(k)(L(k)\tilde{x}(k)) - (V(k+1)(A(k) \\
\quad - \tilde{B}(k)H(k)) + \xi(k)(L(k)\tilde{x}(k) \\
& - \frac{1}{4} V(k+1)\tilde{B}(k) \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \\
& + \gamma(k+1) + D(k)) + (v(k) \\
& + K(k)z(k))' \mathcal{R}_1(v(k) + K(k)z(k)) + \left( \tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) \\
& - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right) \right) \times \mathcal{R}_2 \left( \tilde{u}(k) + \left( H(k) \tilde{\tilde{x}}(k) - \frac{1}{2} \mathcal{R}_2^+ \mathcal{B}(k)' V(k+1)' \right) \right) \}
\]

we get from Equation (52) that Equation (42) is satisfied, completing the proof.

Recalling that \( z(0) = 0 \), we have from Theorem 4.1 that \( PU(v, \xi) = x_0 M(0)x_0 - V(0)x_0 + \gamma(0) \) and, for
\( \mathcal{H}(\omega) = P L_i(\omega), i = 1, 2 \text{ or } 3, \)

\( \mathcal{H}(\omega) = x_0^T M(0)x_0 - V(0)x_0 + \gamma(0), \quad (55) \)

where the input parameters for problems \( PU, PL_1, PL_2 \) and \( PL_3 \) as shown in Table 1. For problems \( PC1, PC2 \) and \( PC3 \), we still have to solve the Lagrangian dual problem \( \max_{\omega \geq 0} \mathcal{H} \) by applying a search algorithm on \( \omega \). As pointed out in Zhu et al. (2004), \( \mathcal{H} \) is a concave function so that a primal-dual method based on the gradient method can be applied. Notice that some extra care need to be taken since at each iteration it is required to check if Assumption 3.1 is true.

Next, we present a sufficient condition for a closed-form solution for the problem \( PC1 \). From Table 1, notice that in this case \( P(k) \) and \( M(k) \), as defined in Equations (28) and (29), will not depend on the parameter \( \omega \), so that Assumption 3.1 can be checked independently of \( \omega \). Notice also that, as seen in Remark 3.2, \( P(k) \geq 0 \) and \( M(k) \geq 0 \) for all \( k \).

Define for \( k = 0, \ldots, T - 1 \),

\[ \bar{A}_H(k) = \bar{A}(k) - \bar{B}(k)H(k), \]

\[ \bar{B}(k, t) = \frac{1}{2} \left( \prod_{j=k+1}^{t-1} \bar{A}_H(j) \right) \bar{B}(k) \mathcal{R}(k, M(k + 1)), \]

and let the elements in row \( r \) and column \( c \) of \( \mathcal{C} \in \mathbb{H}^T \) and \( \mathbb{D} \in \mathbb{H}^{T,1} \) be given as

\[ \mathbb{C}_{rc} = L(r) \sum_{i=0}^{\min(r,c)-1} \bar{B}(i, r) \times \left( \prod_{j=i+1}^{c-1} \bar{A}_H(j) \right) \]

\[ \times (L) \quad \text{and} \quad (58) \]

\[ \mathbb{D}_{r,1} = L(r) \prod_{j=0}^{r-1} \bar{A}_H(j)x_0. \quad (59) \]

The following theorem establishes a sufficient condition for the analytical solution to the Lagrangian dual problem \( PC1 = \max_{\omega \geq 0} \mathcal{H}(\omega), \) where \( \mathcal{H}(\omega) = PL1(\omega) \).

**Theorem 4.2:** Suppose that Assumption 3.1 holds and assume that \( \det(\mathcal{C}) \neq 0 \). Set

\[ \omega^* = \mathcal{C}^{-1}(\mathbb{C} - \mathbb{D}). \quad (60) \]

Then if \( \omega^* \geq 0 \), we have that \( PC1 = \max_{\omega \geq 0} \mathcal{H}(\omega) = \mathcal{H}(\omega^*) \) and an optimal control strategy for problem \( PC1 \) is given by \( u^*(k) = v^*(k) + \bar{u}^*(k) \) as in Equations (43), (44), with the parameter \( \omega = \omega^* \) in Table 1.

**Proof:** Set for any \( u \in \mathbb{U}, \) \( \Psi(u) = \sum_{t=1}^{T} (v(t) \text{ Var}(y^\omega(t))) \), \( \Phi(u) = \sum_{t=1}^{T} (\epsilon(t) - \mathbb{E}(y^\omega(t))) \), so that

\[ \mathcal{H}(\omega) = \min_{\omega \in \mathbb{U}} (\Psi(u) + \omega \Phi(u)). \]

From Theorem 4.1, we have that \( \mathcal{H}(\omega) = \mathbb{U}(\Psi(u^\omega) + \omega \Phi(u^\omega)) \) with \( u^\omega \) as in Equations (43) and (44), where we have replaced the superscript \( * \) by \( \omega \) to indicate the dependence on the parameter \( \omega \). The proof consists in developing Equation (A5) to obtain \( \mathbb{E}(y(t)) \) explicitly on each \( \omega(t) \) and then solving it for \( \omega \) in order to get \( \mathbb{E}(y(t)) = \epsilon(t) \). From Equation (A5) and using the definitions of \( \mathbb{A}_H \) and \( \mathbb{B} \) in Equations (56) and (57), respectively, we have that

\[ \mathbb{E}(y^\omega(t)) = L(t) \prod_{j=0}^{t-1} \bar{A}_H(j)x_0 + L(t) \sum_{i=0}^{t-1} \bar{B}(i, t)V(i + 1). \quad (61) \]

Using Equations (30) and (56), we rewrite \( V(t) \) explicitly on each \( \omega(t) \), \( t = 1, \ldots, T \), as

\[ V(k) = \sum_{c=k}^{T} \left( \prod_{j=k}^{c-1} \bar{A}_H(j) \right) \quad \text{and} \quad (62) \]

Applying Equation (62) into (61), we get that

\[ \mathbb{E}(y^\omega(t)) = L(t) \prod_{j=0}^{t-1} \bar{A}_H(j)x_0 + L(t) \sum_{i=0}^{t-1} \bar{B}(i, t) \]

\[ \times \sum_{c=i+1}^{T} \left( \prod_{j=i+1}^{c-1} \bar{A}_H(j) \right) \quad \text{and} \quad (63) \]

Then we set \( \mathbb{E}(y^\omega(t)) = \epsilon(t) \) for \( t = 1, \ldots, T \), where \( \epsilon(t) \) is a known restriction, and apply Equation (63) to obtain a set of \( T \) equations on
\(T\) unknown \(\omega(t)\). Finally, using the definitions in Equations (58) and (59) for \(r, c = 1, \ldots, T\), we rearrange this system of equations into a vector form as \(\epsilon = C\omega + D\), which can be solved for \(\omega\) as in Equation (60) if \(\det(C) \neq 0\). Set now for any \(u \in \mathbb{U}\), 
\[\Psi(u) = \sum_{t=1}^{T} \nu(t) \text{Var}(y^u(t)), \Phi(u) = \sum_{t=1}^{T} (\epsilon(t) - \mathbb{E}(y^u(t))),\]
so that \(\mathcal{H}(\omega) = \min_{u \in \mathbb{U}} (\Psi(u) + \omega \Phi(u))\). Consider \(u^*\) as in Theorem 4.1 with \(\omega = \omega^*\). We have that 
\[\Phi(u^*) = 0\] and for any \(\omega \geq 0\), \(\mathcal{H}(\omega) \leq \Psi(u^*) + \omega \Phi(u^*) = \Psi(u^*) = \Psi(u^*) + \omega^* \Phi(u^*) = \mathcal{H}(\omega^*)\), so that \(\mathcal{H}(\omega^*) \geq \mathcal{H}(\omega)\) and thus \(\mathcal{P}1 = \max_{\omega \geq 0} \mathcal{H}(\omega) = \mathcal{H}(\omega^*)\), completing the proof. 

\section{5. Examples}

In this section, we apply the results obtained in Section 4 to recover some known results analysed in Cui et al. (2014) for the portfolio selection problem using the mean-field formulation. For that, we write in Section 5.1 the portfolio selection problem as the linear system with multiplicative noise introduced in Section 2.2 and show that the solution derived from Theorem 4.1 coincides with the one obtained in Cui et al. (2014). In Section 5.2, we tackle the portfolio selection problem considering the risk control over the bankruptcy problem. We conclude the section by presenting in Section 5.3 some numerical examples for the portfolio tracking problem using the mean-field formulation presented in Section 4.

\subsection{5.1. Portfolio selection considering problem PU}

Let us consider a financial market with \(n\) risky assets and one riskless asset, with \(e(k) = [e_1(k) \ldots e_n(k)]^\top\) as the vector of random returns of the \(n\) risky assets at period \(k\), and \(s(k)\) as the deterministic return of the riskless asset at period \(k\).

As in Cui et al. (2014), it is assumed that the vectors \(e(k), k = 0, \ldots, T - 1\) are statistically independent, and it is known the vector of first unconditional moments \(\mathbb{E}(e(t)) = [\mathbb{E}(e_1(k)) \ldots \mathbb{E}(e_n(k))]^\top\), and second moments given by the positive definite covariance matrix 
\[\text{cov}(e(k)) = \mathbb{E}(e(k)e(k)^\top) - \mathbb{E}(e(k))\mathbb{E}(e(k))^\top.\]

Define the random vector \(\eta(k) = e(k) - s(k)1\), where \(1\) is the \(n\)-dimensional vector formed by ones in all its entries.

As shown in Cui et al. (2014), \(\text{cov}(\eta(k)) > 0\) and thus \(\mathbb{E}(\eta(k)\eta(k)^\top) > 0\).

Finally set \(B(k) = \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))^{-1}\mathbb{E}(\eta(k))^\top\).

From Lemma 2 in Cui et al. (2014) we have that
\[\text{cov}(e(k))^{-1}\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))^{-1}\mathbb{E}(\eta(k))^\top.\]

Let \(u_i(k)\) represents the amount of wealth allocated to asset \(i\) at time \(k\) with \(i = 1, \ldots, n\). The investment vector strategy \(u(k) = (u_1(k) \ldots u_n(k))^\top\) is said to be admissible if it is \(\mathcal{F}_k\)-measurable, where \(\mathcal{F}_k\) is the \(\sigma\)-field generated by \([e(0), \ldots, e(k-1)], k = 1, \ldots, T\) and \(\mathcal{F}_0\) is the trivial \(\sigma\)-field.

Associated to each admissible investment strategy \(u = [u(0), \ldots, u(T - 1)]\), we have the portfolio’s value process \([x^u(k); t = 0, \ldots, T - 1]\), which represents the investor’s wealth at the end of time \(k\).

For notational simplicity, we shall suppress the superscript \(^u\) whenever no confusion may arise.

Assuming that the initial wealth \(x(0) = x_0 > 0\) and that the portfolio is self-financed, the wealth process is represented by (see, for instance, (Li & Ng, 2000)):
\[x(k + 1) = s(k)x(k) + \eta(k)^\top u(k).\]

Note that the amount of wealth allocated to the riskless asset is determined by \(x(k) - 1^\top u(k)\).

Let us show now that we can write Equation (65) as in Equation (1).

As shown in Proposition 1.1.3 of Davis Vinter (1985), we can write
\[\eta(k) = \mathbb{E}(\eta(k)) + \Gamma(k)w(k), \quad \mathbb{E}(w(k)) = 0, \quad \text{cov}(w(k)) = I,\]
and \(\Gamma(k) = \text{cov}(\eta(k))^{1/2} = \text{cov}(e(k))^{1/2}\). Moreover, since \(\eta(k)\) are statistically independent vectors, we get that \(w(k) = \Gamma(k)^{-1}(\eta(k) - \mathbb{E}(\eta(k)))\) are also statistically independent vectors. Setting \(\Gamma(k) = \begin{bmatrix} \sigma^1(k) & \ldots & \sigma^n(k) \end{bmatrix}\), that is, \(\sigma^j(k)\) is the \(j\)th column of \(\Gamma(k)\), we have from Equation (66) that Equation (65) can be rewritten as
\[x(k + 1) = s(k)x(k) + (\mathbb{E}(\eta(k))' + \sum_{j=1}^{n} \sigma^j(k)^\top w_j(k))u(k)\]
and from Equation (67), we recover Equations (1) and (2) considering \(\hat{A}(k) = s(k), \hat{A}_0 = 0, \hat{B}(k) = \cdots\).
In what follows consider \( l(k) = 0 \) for all \( k = 1, \ldots, T \). Since \( \bar{A}_s(k) = 0 \), we have from Remark 3.1 that \( M(k) = 0 \), and \( A(k, X, Y) = s(k)^2 X, \ G(k, X, Y) = s(k) \mathbb{E}(\eta(k)) X \). From Equation (68),

\[
R(k, X, Y) = \mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))' X + \sum_{j=1}^{n} \sigma^j(k) \sigma^j(k)' Y \\
= \mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))' X + \text{cov}(e(k)) Y,
\]

so that we have

\[
R(k, 0, P(k+1)) = \text{cov}(e(k)) P(k+1) > 0,
\] (69)

provided that \( P(k+1) > 0 \). Noticing that \( \mathbb{E}(\eta(k) \eta(k)') = \text{cov}(\eta(k)) + \mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))' \), we have that for \( Y > 0 \),

\[
P(k, Y) = s(k)^2 Y - s(k)^2 Y \mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))' \mathbb{E}(\eta(k))^{-1} \mathbb{E}(\eta(k)) + v(k) \\
= s(k)^2 Y (1 - \mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))^{-1} \mathbb{E}(\eta(k)) + v(k) \\
= s(k)^2 Y (1 - B(k)) + v(k).
\] (70)

From this and Equation (28), we get that \( P(k) > 0 \) and \( P(k) = s(k)^2 (1 - B(k)) P(k) + v(k) \), \( k = 0, \ldots, T - 1 \), \( P(T) = v(T) \).

From Equation (69), we have that Assumption 3.1 holds true.

Note now that, since \( G(k, M(k+1), P(k+1)) = G(k, 0, P(k+1)) = 0 \), we have from Equation (30) that

\[
V(k) = V(k, M(k+1), P(k+1), V(k+1)) \\
= s(k) V(k+1) + \xi(k), \quad V(T) = \xi(T),
\]

and since from Equation (64),

\[
\bar{B}(k) R(k, M(k+1), P(k+1)) \bar{B}(k)' \\
= \frac{1}{P(k+1)} \mathbb{E}(\eta(k))' \text{cov}(e(k))^{-1} \mathbb{E}(\eta(k))
\]

we get from Equations (25) and (31) that for \( k = T - 1, \ldots, 0 \),

\[
\gamma(k) = \gamma(k+1) \\
- \frac{V(k+1)^2}{4P(k+1) (1 - B(k))}, \quad \gamma(T) = 0.
\]

Repeating the arguments above, we have from Equation (32) that

\[
K(k) = s(k) (\mathbb{E}(\eta(k)) \mathbb{E}(\eta(k))' + \text{cov}(e(k)))^{-1} \mathbb{E}(\eta(k)) \\
= s(k) \mathbb{E}(\eta(k) \eta(k)')^{-1} \mathbb{E}(\eta(k))
\]

and from Equation (33) that \( H(k) = 0 \). From Equations (43), (44) and (64), we get that

\[
v^*(k) = -s(k) \mathbb{E}(\eta(k) \eta(k)')^{-1} \\
\times \mathbb{E}(\eta(k))(x(k) - \mathbb{E}(x(k))),
\]

\[
\tilde{u}^*(k) = \frac{V(k+1)}{2P(k+1)} \text{cov}(e(k))^{-1} \mathbb{E}(\eta(k)) \\
= \left( \frac{V(k+1)}{2P(k+1)} \right) \mathbb{E}(\eta(k) \eta(k)')^{-1} \mathbb{E}(\eta(k))
\]

and from Equation (42), we obtain that \( f_k(\mathbb{E}(x(k)), x(k) - \mathbb{E}(x(k))) = P(k)(x(k) - \mathbb{E}(x(k)))^2 - V(k) \mathbb{E}(x(k)) + \gamma(k) \).

Finally, for problem \( PU(v, \xi) \), we have that

\[
\bar{A}(k) - \bar{B}(k) H(k) = s(k) - \mathbb{E}(\eta(k))' 0 = s(k)
\] (71)

and

\[
\frac{1}{2} \bar{B}(k) R^\dagger(k) \bar{B}(k)' V(k+1) \\
= \mathbb{E}(\eta(k))' \text{cov}(e(k))^{-1} \mathbb{E}(\eta(k)) V(k+1) \\
- \frac{1}{2P(k+1)} \mathbb{E}(\eta(k))' \mathbb{E}(\eta(k) \eta(k)')^{-1} \mathbb{E}(\eta(k)) V(k+1) \\
= \frac{B(k) V(k+1)}{2(1 - B(k)) P(k+1)}.
\] (72)

Applying Equations (71) and (72) into Equation (A5), we obtain that

\[
\mathbb{E}\left( y^\mu(t) \right) = x_0 \prod_{j=0}^{t-1} s(j) + \sum_{i=0}^{t-1} \left( \prod_{j=i+1}^{t-1} s(j) \right)
\]
These results coincide with those obtained in Proposition 1 in Cui et al. (2014).

5.2. Portfolio selection considering the risk control over bankruptcy

We now apply the results regarding the mean-variance with risk-control over a minimum expected output obtained in Section 4 to recover some known results analysed in Cui et al. (2014) using the mean-field formulation. Let us consider a financial market as defined in Section 5.1 and, similarly as in Cui et al. (2014), a modification of problem PC3(ω) stated as

\[
PC3(\xi, a, b) : \max_{u \in U} \left( \xi(t)E\left( y^u(t) \right) - \omega(t)Var\left( y^u(t) \right) \right) \tag{73}
\]

s.t. \( Var\left( y^u(t) \right) \leq a(t) \left[ E\left( y^u(t) \right) - b(t) \right]^2 \). \tag{74}

Taking \( L(t) = 1, \xi(t) = 0, t = 1, \ldots, T - 1, \xi(T) = 1, a(T) = 0 \) and \( \xi_\omega(T) = 1 \), we get the problem as defined in Equation (11) for the Lagrangian multipliers \( \omega(t), t = 1, \ldots, T - 1 \). Thus, we have that \( P(T) = \omega(T), M(T) = 0, V(T) = 1, \) and \( \gamma(T) = 0. \) Since \( \tilde{A}_k(\omega) = 0 \), we have that \( \mathcal{A}(k, X, Y) = s(k)^2X \) and \( \mathcal{G}(k, X, Y) = s(k)E(\eta(k))X \). From Equations (28) and (70), we have that \( P(k) = s(k)^2(1 - B(k))P(k + 1) + \omega(k), k = 0, \ldots, T - 1, P(T) = \omega(T), \) and thus \( P(k) > 0 \). From Equation (68), we have that

\[
\mathcal{R}(k, X, Y) = E(\eta(k))E(\eta(k))'X + \sum_{j=1}^{n} \sigma^j(k)\sigma^j(k)'Y
\]

\[
= E(\eta(k))E(\eta(k))'X + cov(e(k))Y
\]

\[
= E(\eta(k))E(\eta(k))'Y - E(\eta(k))E(\eta(k))'
\times (Y - X). \tag{75}
\]

Applying Lemma 4 in Cui et al. (2014), we have that

\[
\mathcal{R}(k, M(k + 1), P(k + 1))^{-1}E(\eta(k)) = \frac{E(\eta(k))E(\eta(k))^{-1}E(\eta(k))}{B(k)M(k + 1) + (1 - B(k))P(k + 1)} \tag{76}
\]

provided that \( B(k)M(k + 1) + (1 - B(k))P(k + 1) \neq 0. \)

Define

\[
\delta(k + 1) = \frac{(1 - B(k))P(k + 1)}{B(k)M(k + 1) + (1 - B(k))P(k + 1)}. \tag{77}
\]

Since \( B(k) = \mathbb{E}(\eta(k))'\mathbb{E}(\eta(k))\eta(k)E(\eta(k)) \), we have from Equations (29) and (76),

\[
M(k) = \mathcal{A}(k, X, Y) - \mathcal{G}(k, X, Y)'\mathcal{R}(k, X, Y)^{-1}\mathcal{G}(k, X, Y) + \omega(k)a(k)L(k)'L(k)
\]

\[
= s(k)^2M(k + 1)
\]

\[
\frac{s(k)^2M(k + 1)^2\mathbb{E}(\eta(k))}{\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))^{-1}\mathbb{E}(\eta(k))}
\]

\[
= \frac{B(k)M(k + 1) + (1 - B(k))P(k + 1)}{B(k)M(k + 1) + (1 - B(k))P(k + 1)}
\]

\[
- \omega(k)a(k)
\]

\[
= s(k)^2\delta(k + 1)M(k + 1)
\]

\[
- \omega(k)a(k). \tag{77}
\]

From Proposition A.2 in the Appendix, we have that if \( B(k)M(k + 1) + (1 - B(k))P(k + 1) > 0 \) then Assumption 3.1 will hold and \( M(k) \) is given by (77).

From Equation (30),

\[
V(k) = V(k + 1) \left( s(k) - \frac{s(k)B(k)M(k + 1)\mathbb{E}(\eta(k))'}{B(k)M(k + 1) + (1 - B(k))P(k + 1)} \right) - \omega(k)a(k)b(k)
\]

\[
= V(k + 1) \left( s(k)B(k)M(k + 1) + s(k)(1 - B(k)) \times P(k + 1) - s(k)B(k)M(k + 1) \right)
\]

\[
= V(k + 1) \left( \frac{B(k)M(k + 1) + (1 - B(k))P(k + 1)}{B(k)M(k + 1) + (1 - B(k))P(k + 1)} \right) - \omega(k)a(k)b(k)
\]

\[
= s(k)\delta(k + 1)V(k + 1) - \omega(k)a(k)b(k). \tag{78}
\]
and from Equation (31),
\[
\gamma(k) = \gamma(k + 1) + \frac{V(k + 1)^2 \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \times \mathbb{E}(\eta(k))}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)} + \omega(k) a(k) b(k) \gamma(k + 1)
\]
\[
+ \frac{V(k + 1)^2 B(k)}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)} + \omega(k) a(k) b(k)^2.
\]
(79)

Note that repeating the arguments above, we have from Equation (32) that
\[
K(k) = s(k) \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k))
\]
(80)

and from Equation (33) that
\[
H(k) = \frac{s(k)M(k + 1) \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k))}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)}.
\]
(81)

From Equations (43) and (44), we get that
\[
\nu^*(k) = -s(k) \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k)) z(k),
\]
\[
\tilde{u}^*(k) = -\frac{s(k)M(k + 1) \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k))}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)} \tilde{x}(k)
\]
\[
+ \frac{V(k + 1) \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k))}{2B(k)M(k + 1) + (1 - B(k)) P(k + 1)}
\]
\[
+ 0.5 \frac{V(k + 1) - s(k)M(k + 1) \tilde{x}(k)}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)} \times \mathbb{E}(\eta(k)\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k)),
\]
and from Equation (55), we obtain that
\[
\mathcal{H}(\omega) = M(1) \delta(1) s(0)^2 x(0)^2 - V(1) \delta(1) s(0) x(0)
\]
\[
- \sum_{j=0}^{t-1} \left[ \frac{V(j + 1)^2 B(j)}{B(j)M(j + 1) + (1 - B(j)) P(j + 1)} \right. \right.
\]
\[
+ \omega(j) a(j) b(j)^2 \left. \right].
\]
 Finally, we apply Equations (A5), (A6), and the operators in Equation (A4) to recover the expected output and its variance formulas obtained in Cui et al. (2014). For problem PL3(\omega), we have that
\[
\tilde{A}(k) - \tilde{B}(k) H(k)
\]
\[
= s(k) - \frac{s(k)M(k + 1) \mathbb{E}(\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k))}{B(k)M(k + 1) + (1 - B(k)) P(k + 1)}
\]
\[
= \frac{B(k)M(k + 1) + (1 - B(k)) P(k + 1)}{s(k)}
\]
(82)

and
\[
\frac{1}{2} \tilde{B}(k) R^*(k) \tilde{B}(k)' V(k + 1)
\]
\[
= \frac{\mathbb{E}(\eta(k')\eta(k)\eta(k')^{-1}) \mathbb{E}(\eta(k)) V(k + 1)}{2(B(k)M(k + 1) + (1 - B(k)) P(k + 1))}
\]
\[
= \frac{B(k) V(k + 1)}{2(B(k)M(k + 1) + (1 - B(k)) P(k + 1))}.
\]
(83)

Applying Equations (82) and (83) into Equation (A5), we obtain that
\[
\mathbb{E} \left( \gamma^\mu(t) \right)
\]
\[
= x_0 \prod_{j=0}^{t-1} \delta(j + 1) s(j)
\]
\[
+ \sum_{i=0}^{t-1} \left( \prod_{j=i}^{t-1} \delta(j + 1) s(j) \right) \times \frac{B(i) V(i + 1)}{2(B(i)M(i + 1) + (1 - B(i)) P(i + 1))}.
\]

From the Equations (A4) and (A6), we obtain that
\[
\text{Var} \left( \gamma^\mu(t) \right)
\]
\[
= \sum_{j=0}^{t-1} \left[ \frac{(0.5 V(j + 1) - s(j) M(j + 1) \tilde{x}(j)^2 (B(j) - B(j)^2)}{(B(j) M(j + 1) + (1 - B(j)) P(j + 1))^2} \right.
\]
\[
+ \prod_{l=j+1}^{t-1} s(l) \tilde{x}(l)^2 \right].
\]

These results coincide with those obtained in Section IV in Cui et al. (2014).
5.3. Numerical results for a portfolio selection problem with a benchmark

In this section, we apply our results to the selection of a portfolio with one riskless asset and \( n \) risk assets against a benchmark.

Using the same definitions of \( e(k) \), \( \eta(k) \) and \( u(k) \) as in Section 5.1, we recover Equations (1) and (2) by defining the random return of the benchmark as \( r_b(k) \), its expected return as \( b(k) = \mathbb{E}(r_b(k)) \), and the random vector

\[
\hat{\eta}(k) = \begin{bmatrix} \eta(k) \\ r_b(k) \end{bmatrix} = \begin{bmatrix} \mathbb{E}(\eta(k)) \\ b(k) \end{bmatrix} + \hat{\Gamma}(k)w(k),
\]

\( \mathbb{E}(w(k)) = 0, \quad \text{cov}(w(k)) = I, \) \hspace{1cm} (84)

where

\[
\hat{\Gamma}(k)\hat{\Gamma}'(k) = \text{cov}(\hat{\eta}(k)) = \begin{bmatrix} \text{cov}(\eta(k)) & \text{cov}(\eta(k), r_b(k))' \\ \text{cov}(\eta(k), r_b(k)) & \text{cov}(r_b(k)) \end{bmatrix}.
\]

Defining the portfolio's value process as \( X_1(k) \), the benchmark process as \( X_2(k) \), and assuming that \( X_1(0) = X_2(0) = X_0 > 0 \) and that the portfolio is self-financed, we have that

\[
X_1(k + 1) = s(k)X_1(k) + \eta(k)'u(k) \quad \text{and} \quad (85)
\]

\[
X_2(k + 1) = r_b(k)X_2(k) \quad \text{and} \quad (86)
\]

represent the wealth and benchmark processes (see, for instance, Li and Ng (2000)). Setting \( \hat{\Gamma}(k) = [\hat{\sigma}^1(k) \ldots \hat{\sigma}^{n+1}(k)] \), that is, \( \hat{\sigma}^j(k) \) is the \( j \)th column of \( \hat{\Gamma}(k) \), with \( \hat{\sigma}^j(k) = [\hat{\sigma}^j(k)' \hat{\sigma}^j(k)'] \), and noticing that

\[
\hat{\Gamma}(k)\hat{\Gamma}'(k) = \sum_{j=1}^{n+1} \hat{\sigma}^j(k)\hat{\sigma}^j(k)'
\]

\[
= \begin{bmatrix} \hat{\sigma}^1(k) & \ldots & \hat{\sigma}^{n+1}(k) \\ \hat{\sigma}^1(k)' & \cdots & \hat{\sigma}^{n+1}(k)' \\ & \ddots & \vdots \\ & & \hat{\sigma}^{n+1}(k)' \end{bmatrix}
\]

we have from Equation (84) that Equations (85) and (86) can be rewritten as

\[
X_1(k + 1) = s(k)X_1(k) + \mathbb{E}(\eta(k))' + \sum_{j=1}^{n+1} \sigma^j(k)w_j(k)\]

\times u(k) \quad \text{and} \quad \text{cov}(\eta(k))' = \begin{bmatrix} & \tilde{\sigma}^j(k) \end{bmatrix},
\]

\[
X_2(k + 1) = \left( b(k) + \sum_{j=1}^{n+1} \sigma^j_b(k)w_j(k) \right) X_2(k). \quad (88)
\]

Finally, we obtain Equations (1) and (2) by considering Equations (87) and (88) with

\[
x(k) = \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}, \quad \tilde{A}(k) = \begin{bmatrix} s(k) & 0 \\ 0 & b(k) \end{bmatrix},
\]

\[
\tilde{B}(t) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^j_b(k) \end{bmatrix}, \quad L(k) = [1 \quad -1],
\]

and \( \rho_{s_1,s_2}(k) = 1 \) for \( s_1 = s_2 = 1 \) and 0 otherwise.

For our numerical example, we consider the data in Li and Ng (2000) with the annual returns given by \( s(k) = 1.04 \) and \( \mathbb{E}(\eta(k))' = [1.162 \ 1.246 \ 1.228] \). The covariance of the risk assets is

\[
\sigma(k)\sigma(k)' = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix},
\]

\( k = 0, \ldots, T - 1 \).

We also assume a riskless benchmark with \( b(k) = 1.20 \).

We solve problems \( PU, PC1, PC2 \) and \( PC3 \) by applying Theorem 4.1 together with Table 1. For all four problems, we set \( x_0 = [1.0 \ 1.0]' \), a time horizon of \( T = 5 \) weeks, the risk coefficients and their respective restrictions as in Table 2.

To solve problem \( PU(v, \xi) \), we start by computing backwards the operators in Equations (23) and (24) using the definitions as in Equations (28)–(30). Then, using Equations (32) and (33), we can compute \( v^*(k) \) and \( \tilde{v}^*(k), k = 0, \ldots, T - 1 \), applying Equations (43) and (44). Finally, the expected output and variance is calculated using Proposition A.3, leading to \( \mathbb{E}(y^{u^*_T}) = [0.0088, 0.0174, 0.0260, 0.0344, 0.0427] \) and \( \text{Var}(y^{u^*_T}) = [0.0060, 0.0119, 0.0173, 0.0226, 0.0277] \).

Table 2. Risk and restrictions coefficients.

<table>
<thead>
<tr>
<th>Problem</th>
<th>( v(k) )</th>
<th>( \xi(k) )</th>
<th>Restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( PU )</td>
<td>1</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>( PL1 )</td>
<td>1</td>
<td>( \omega(k) )</td>
<td>( \epsilon ) = [0.02, 0.02, 0.03, 0.05, 0.08]'</td>
</tr>
<tr>
<td>( PL2 )</td>
<td>( \omega(k) )</td>
<td>1</td>
<td>( \varphi ) = [0.005, 0.01, 0.01, 0.015, 0.015]'</td>
</tr>
<tr>
<td>( PL3 )</td>
<td>( \omega(k) )</td>
<td>1</td>
<td>( \alpha(t) = 0.05 ) and ( \beta(t) = 0.1 )</td>
</tr>
</tbody>
</table>
Table 3. Lagrangian multipliers.

<table>
<thead>
<tr>
<th>Problem</th>
<th>(\omega^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PL1</td>
<td>[8.909, 0, 0, 1.110, 2.960]</td>
</tr>
<tr>
<td>PL2</td>
<td>[1.181, 0, 13.700, 0, 4.276]</td>
</tr>
<tr>
<td>PL3</td>
<td>[0, 13.062, 12.531, 12.018, 11.522]</td>
</tr>
</tbody>
</table>

In order to compare our results, we also solve \(PU\) using an embedding scheme as applied in Costa and Oliveira (2012), where an auxiliary problem parameterised in \(\lambda\) is solved. This technique leads to the same optimal control law, expected output and variance as before, corroborating our formulation.

In problems \(PC1\), \(PC2\) and \(PC3\), we have to solve the Lagrangian dual problem \(PC_i = \max_{\omega \geq 0} \mathcal{H}(\omega)\), where \(\mathcal{H}(\omega) = PL_i(\omega)\), \(i = 1, 2, 3\) is given by Equation (55) with their respective input parameters as in Table 1. In this paper, we adopt the Nelder-Mead simplex method to solve the Lagrangian problems, which is an available option of the Python optimisation function ‘scipy.optimize.sco.fmin’. The resulting Lagrangian multipliers for each problem are shown in Table 3.

In the case of problem \(PC1(\omega)\), we can also obtain \(\omega^*\) analytically.

Thus, applying Theorem 4.2, we obtain that

\[
C = \begin{bmatrix}
2.305 & 2.306 & 2.308 & 2.310 & 2.312 \\
2.306 & 5.160 & 5.164 & 5.168 & 5.1720 \\
2.308 & 5.164 & 8.933 & 8.940 & 8.946 \\
2.312 & 5.1720 & 8.946 & 14.548 & 25.626 \\
\end{bmatrix} \times 10^{-3},
\]

\(det(C) = 1.53 \times 10^{-12}\), and \(\mathbb{D} = [-2.758, -5.527, -8.3092, -11.102, -13.907]' \times 10^{-3}\).

Finally, applying Equation (60), we get the same \(\omega^*\) as in Table 3 for \(PL1\), corroborating our results.

6. Conclusion

In this paper, we have considered the stochastic mean-variance optimal control problem of discrete-time linear systems with multiplicative noises by adopting the mean-field formulation.

We applied this method to models with no constraints, with intertemporal restrictions on either the expected value of the output or its variance and with restrictions on the minimum value of the output associated with a given probability of occurrence. An explicit sufficient condition for the existence of an optimal control strategy for a general unconstrained problem and the value functions for the dual Lagrangian optimisation problems for the constrained cases were derived. For one of the constrained problems, a sufficient condition for an explicit solution was also presented. The optimal control law is written as a state feedback added with a deterministic sequence, with the input parameters depending on the problem under consideration (see Theorem 4.1).

The solution is derived from a generalised Riccati difference equation interconnected with a set of linear recursive equations (see the definitions of \(P(k), M(k), V(k)\) in Equations (28), (29) and (30)).

When specialised to the optimal portfolio selection problem, we showed that our results retrieve some known results in the literature.

We also applied our formulation to a numerical case of a multi-period portfolio selection problem with a benchmark, where we find the best asset allocation to minimise the sum of the trade-off between the variance and the excess return of the portfolio against a benchmark.

Future works would consider adding constraints to the controls or even studying our problems under the mean-field game theory when two or more population of agents compete instead of cooperating (mean-field control). For instance, we would apply mean-field games to problems with restrictions or with mean-variance functional costs rather than quadratic costs, see (Moon, 2019; Wang et al., 2019) for examples in the continuous-time case.

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References


In this appendix, we present several auxiliary results used in the paper. We start by recalling the following result known as the Schur’s complement.

**Proposition A.1 (Schur’s complement):** Suppose that $Q > 0$ and $R > 0$. The following assertions are equivalent.

(a) $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \succeq 0$.
(b) $Q \succeq SR^{-1}S'$.
(c) $R \succeq SQ^{-1}S'$.

Next we present the following auxiliary result.

**Proposition A.2:** For given scalars $X$ and $Y > 0$, if $B(k)X + (1 - B(k))Y > 0$ then

$\mathcal{R}(k, X, Y) = \mathbb{E}(\eta(k)\eta(k)') Y + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))' (X - Y) > 0$. (A1)

**Proof:** From $B(k)X + (1 - B(k))Y > 0$, we have that $X - Y > -Y/B(k)$, and thus

$\mathbb{E}(\eta(k)\eta(k)') Y + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))' (X - Y)$
\[
\begin{align*}
\mathbb{E}(\eta(k)\eta'') &= \left(\mathbb{E}(\eta(k)\eta'') - \frac{\mathbb{E}(\eta(k)\eta''')}{\mathbb{E}(\eta(k))}\right) Y. \quad (A2)
\end{align*}
\]

By definition, we have that \( B(k) = \mathbb{E}(\eta(k)')/\mathbb{E}(\eta(k)) \), and thus, from the Schur's complement (Proposition A.1), we have that
\[
\begin{align*}
\left[ \begin{array}{cc}
\mathbb{E}(\eta(k)\eta(k')) & \mathbb{E}(\eta(k)) \\
\mathbb{E}(\eta(k')) & B(k)
\end{array} \right] &\geq 0 \\
\Leftrightarrow \left( \mathbb{E}(\eta(k)\eta(k')) - \frac{\mathbb{E}(\eta(k)\eta''')}{\mathbb{E}(\eta(k))} \right) &\geq 0. \quad (A3)
\end{align*}
\]

Applying Lemma 4 in Cui et al. (2014), we have that \( \mathcal{R}(k, X, Y) \) has an inverse since by assumption \( B(k)X + (1 - B(k))Y > 0 \). From this, (A3) and (A2), we get (A1).

For \( k = 0, \ldots, T - 1 \) and \( j = 1, \ldots, T \), the following new operators will be useful to present an expression for the output variance \( \text{Var}(y^u(t)) \) when the optimal control strategy \( u^*(k) = v^*(k) + \bar{u}^*(k) \) is applied to system (1). For \( Y \in \mathbb{R}^p \), define
\[
\begin{align*}
\tilde{P}(k, Y) &= \mathcal{A}(k, Y, Y) + K(k)'\mathcal{R}(k, Y, Y)K(k) \\
&\quad - 2\mathcal{G}(k, Y)'K(k), \\
\tilde{Q}(k, Y) &= \mathcal{A}(k, 0, Y) + H(k)'\mathcal{R}(k, 0, Y)H(k) \\
&\quad - 2\mathcal{G}(k, 0)'H(k), \\
\tilde{H}(k, Y) &= \mathcal{G}(k, 0, Y)'\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\tilde{B}(k)'V(k+1)' \\
&\quad - H(k)'\mathcal{R}(k, 0, Y)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\tilde{B}(k)'V(k+1)', \\
\tilde{S}(k, Y) &= \frac{1}{4}V(k+1)\tilde{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\mathcal{R}(k, 0, Y) \\
&\quad \times \mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\tilde{B}(k)'V(k+1)', \\
\theta^j_k &= \tilde{P}(j, \ldots, \tilde{P}(k-1, \tilde{P}(k, L(k+1)'L(k+1))) \ldots), \\
\theta^j_k &= L(k+1)'L(k+1) \quad \text{if } j > k. \quad (A4)
\end{align*}
\]

We have the following proposition.

**Proposition A.3:** Suppose that Assumption 3.1 holds. If the optimal control strategy \( u^*(k) = v^*(k) + \bar{u}^*(k) \) as in (43) and (44) is applied to system (1), then the expected value of the output \( \mathbb{E}(y^u(t)) \) and the variance output \( \text{Var}(y^u(t)) \) are given respectively by
\[
\begin{align*}
\mathbb{E}(y^u(t)) &= L(t) \prod_{j=0}^{t-1} \left( \tilde{A}(j) - \tilde{B}(j)H(j) \right) x_0 + L(t) \\
&\quad \times \sum_{i=0}^{t-1} \left\{ \left( \prod_{j=1}^{t-1} \left( \tilde{A}(j) - \tilde{B}(j)H(j) \right) \right) \\
&\quad \times \left( \frac{1}{2} \tilde{B}(i)\mathcal{R}(i, M(i+1), P(i+1))^{\dagger} \right) \right\} \\
&\quad \times \left[ \tilde{B}(i)'V(i+1) \right] \\
\text{Var}(y^u(t)) &= \sum_{j=0}^{t-1} \left[ \tilde{x}(j)'\tilde{Q}(j, \theta^j_{j+1}^{-1})\tilde{x}(j) \\
&\quad + \tilde{x}(j)'\tilde{R}(j, \theta^j_{j+1}^{-1}) + \tilde{S}(j, \theta^j_{j+1}^{-1}) \right]. \quad (A6)
\end{align*}
\]

**Proof:** To ease the notation, we remove the superscript dependence on \( u^* \). Substituting (44) into (12) we obtain that
\[
\begin{align*}
\tilde{x}(k+1) &= (\tilde{A}(k) - \tilde{B}(k)H(k))\tilde{x}(k) \\
&\quad + \frac{1}{2}\tilde{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\tilde{B}(k)'V(k+1)'. \\
&\quad (A7)
\end{align*}
\]

Iterating (A7) for \( k = 0, \ldots, T-1 \), with \( x(0) = x_0 \), and using (2), we obtain (A5). To prove Equation (A6), we use the dynamics in Equation (13), the independence hypothesis made on the multiplicative noises, and recall that \( \text{Var}(y^u(t)) = \mathbb{E}((L(t)z(t))^2) \). To ease the notation, we set \( Y = L(k+1)'L(k+1) \). Thus, for \( t = k + 1 \), we have that
\[
\begin{align*}
\mathbb{E}(z(k+1)'Yz(k+1)) &= \mathbb{E} \left\{ z(k)' \left[ \mathcal{A}(k, Y, Y) \right] z(k) \right\} \\
&\quad + \tilde{x}(k)' \left[ \sum_{s=1}^{e} \tilde{A}_s(k)Y\tilde{A}_s(k) \right] \tilde{x}(k) \\
&\quad + v(k)' \left[ \mathcal{R}(k, Y, Y) \right] v(k) + u(k)' \\
&\quad \times \left[ \sum_{s=1}^{e} \tilde{B}_s(k)Y\tilde{B}_s(k) \right] u(k) + 2z(k)' \left[ \mathcal{G}(k, Y, Y) \right] v(k) \\
&\quad + 2\tilde{x}(k)' \left[ \sum_{s_1,s_2=1}^{e} \rho_{s_1,s_2}(k)\tilde{A}_{s_1}(k)'Y\tilde{B}_{s_2}(k) \right] \tilde{u}(k) + G(k), \\
&\quad (A8)
\end{align*}
\]

where
\[
\begin{align*}
G(k) &= 2\mathbb{E}(z(k))' \left[ \sum_{s_1=1}^{e} \tilde{A}_s(k)Y\tilde{A}_s(k) \right] \tilde{x}(k) \\
&\quad + 2\mathbb{E}(z(k))' \left[ \sum_{s_1,s_2=1}^{e} \rho_{s_1,s_2}(k)\tilde{A}_{s_1}(k)'Y\tilde{B}_{s_2}(k) \right] \tilde{u}(k) \\
&\quad + 2\tilde{x}(k)' \left[ \sum_{s_1,s_2=1}^{e} \rho_{s_1,s_2}(k)\tilde{A}_{s_1}(k)'Y\tilde{B}_{s_2}(k) \right] \mathbb{E}(v(k)) \\
&\quad + 2\mathbb{E}(v(k))' \left[ \sum_{s_1=1}^{e} \tilde{B}_s(k)'Y\tilde{B}_s(k) \right] \tilde{u}(k).
\end{align*}
\]

Note that \( G(k) = 0 \) since \( \mathbb{E}(z(k)) = 0 \) and \( \mathbb{E}(v(k)) = 0 \). Therefore, applying Equations (43) and (44) into (A8), we obtain that
\[
\begin{align*}
\mathbb{E}(z(k+1)'Yz(k+1)) &= \mathbb{E} \left\{ z(k)' \left[ \mathcal{A}(k, Y, Y) \right] z(k) \right\} \\
&\quad + \tilde{x}(k)' \left[ \mathcal{A}(k, 0, Y) \right] \tilde{x}(k) \\
&\quad + z(k)' \left[ \mathcal{K}(k)'\mathcal{R}(k, Y, Y)K(k) \right] z(k) + \tilde{x}(k)' \\
&\quad \times \left[ \mathcal{H}(k)'\mathcal{R}(k, 0, Y)H(k) \right] \tilde{x}(k) \\
&\quad + \frac{1}{4}V(k+1)\tilde{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}
\end{align*}
\]
Rearranging the terms in Equation (A9) and applying the operators (A4), we obtain that

\[ E(z(k + 1)' Yz(k + 1)) = E \left\{ z(k)' \bar{P}(k, Y) z(k) \right\} + \bar{x}(k)' \bar{Q}(k, Y) \bar{x}(k) + \bar{x}(k)' \bar{R}(k, Y) + \bar{S}(k, Y). \]  

Finally, we apply Equation (A10) recursively on \( E[z(k)' \bar{P}(k, Y) z(k)] \) and so on to obtain Equation (A6), completing the proof.